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A Nonstandard Supersymmetric KP Hierarchy

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Abstract

We show that the supersymmetric nonlinear Schrödinger equation can be written as a constrained super KP flow in a nonstandard representation of the Lax equation. We construct the conserved charges and show that this system reduces to the super mKdV equation with appropriate identifications. We construct various flows generated by the general nonstandard super Lax equation and show that they contain both the KP and mKP flows in the bosonic limits. This nonstandard supersymmetric KP hierarchy allows us to construct a new super KP equation which is nonlocal.

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1. Introduction

Integrable models have been studied vigorously in recent years from various points of view [1-3]. In particular, we note that various two dimensional gravity theories and continuum string equations arise naturally from the study of such systems. Even matrix models, in their continuum limit, contain such systems and this has led to a lot of interest in their study (see [4] and references therein). These are models in $1+1$ or $2+1$ dimensions which are most commonly represented in terms of a Lax operator which is a pseudo-differential operator of the general form [5]

$$L = \partial^n + u_{-1}\partial^{n-1} + u_0\partial^{n-2} + \cdots + u_{n-3}\partial^{-1} + \cdots \quad (1.1)$$

Here $u_i(x)$'s are dynamical variables and their time evolution is given in terms of a Lax equation which in the standard representation has the form

$$\frac{\partial L}{\partial t_k} = \left[L, (L^{k/n})_+ \right] = - \left[L, (L^{k/n})_- \right] \quad (1.2)$$

where $(L^{k/n})_-$ ($(L^{k/n})_+$) denotes the part of the pseudo-differential operator containing only negative (nonnegative) powers of ∂ . Most integrable models studied in $1+1$ and $2+1$ dimensions have a standard Lax representation of the form in (1.2).

It is known, by now, that there exist integrable models which have a nonstandard Lax representation of the form

$$\frac{\partial L}{\partial t_k} = \left[L, (L^{k/n})_{\geq 1} \right] \quad (1.3)$$

where $(\)_{\geq 1}$ represents the projection onto the purely differential part of a pseudo-differential operator. The dispersive long water wave equation [6] or equivalently the two boson hierarchy [7-11] has been studied from this point of view and this in turn has led to the study of constrained KP hierarchies [12]. However, not much is known about the properties of the supersymmetric generalizations of such system. In a recent paper [13], we studied the supersymmetrization of the two boson hierarchy and showed how it gives the supersymmetric nonlinear Schrödinger (NLS) equation [14,15] with appropriate field redefinitions. In the present paper we report on further general results in the study of nonstandard supersymmetric Lax systems.

In sec. 2 we review briefly known results on the formulation of the nonlinear Schrödinger equation as a constrained KP system with our observations that become useful in the later sections. In sec. 3, we shown how the supersymmetric nonlinear Schrödinger equation can be written as a constrained super KP system but with a nonstandard Lax representation. We construct the conserved charges and one of the Hamiltonian structures associated with this system. We also show how the supersymmetric mKdV equation can be embedded into this system with appropriate field identifications. In sec. 4 we study various flows associated with a supersymmetric nonstandard KP system. We show that in the bosonic limit, this system contains both the KP as well as the mKP flows. This allows us to construct in sec. 5 a new supersymmetric KP equation which is nonlocal. It, however, leads upon reduction to the supersymmetric KdV equation. We present our conclusions in sec. 6.

2. NSE As a Constrained KP System

The two boson hierarchy is represented by a Lax operator of the form [6,11]

$$L = \partial - J_0 + \partial^{-1} J_1 \quad (2.1)$$

and the nonstandard Lax equation

$$\frac{\partial L}{\partial t} = \left[L, (L^2)_{\geq 1} \right] \quad (2.2)$$

leads to the system of integrable equations

$$\begin{aligned} \frac{\partial J_0}{\partial t} &= (2J_1 + J_0^2 - J_0')' \\ \frac{\partial J_1}{\partial t} &= (2J_0 J_1 + J_1')' \end{aligned} \quad (2.3)$$

where a prime denotes differentiation with respect to x . It is now straight forward to check that with the field identifications [7-10]

$$\begin{aligned} J_0 &= -\frac{q'}{q} = -(\ln q)' \\ J_1 &= \bar{q}q \end{aligned} \quad (2.4)$$

the system of equations in (2.3) reduce to the nonlinear Schrödinger equation

$$\begin{aligned}\frac{\partial q}{\partial t} &= -(q'' + 2(\bar{q}q)q) \\ \frac{\partial \bar{q}}{\partial t} &= \bar{q}'' + 2(\bar{q}q)\bar{q}\end{aligned}\tag{2.5}$$

Let us next consider the Lax operator (2.1) with the field identifications in (2.4) and note that

$$\begin{aligned}L &= \partial + \frac{q'}{q} + \partial^{-1}\bar{q}q \\ &= q^{-1}(\partial + q\partial^{-1}\bar{q})q \\ &= G\tilde{L}G^{-1}\end{aligned}\tag{2.6}$$

where

$$\begin{aligned}G &= q^{-1} \\ \tilde{L} &= \partial + q\partial^{-1}\bar{q}\end{aligned}\tag{2.7}$$

The two Lax operators, L and \tilde{L} , are said to be related through a gauge transformation [7,10,16]. However, it can be easily checked that in terms of the Lax operator \tilde{L} , the nonlinear Schrödinger equation can be written in the standard Lax representation

$$\frac{\partial \tilde{L}}{\partial t} = [\tilde{L}, (\tilde{L}^2)_+]\tag{2.8}$$

Let us note that the Lax operator, \tilde{L} , in (2.7) can also be written as

$$\begin{aligned}\tilde{L} &= \partial + q\bar{q}\partial^{-1} - q\bar{q}'\partial^{-2} + q\bar{q}''\partial^{-3} + \dots \\ &= \partial + \sum_{n=0}^{\infty} u_n \partial^{-n-1}\end{aligned}\tag{2.9}$$

with

$$u_n = (-1)^n q\bar{q}^{(n)}\tag{2.10}$$

Here $f^{(n)}$ represents the n th derivative with respect to x . Note that the form of \tilde{L} in the last expression in (2.9) is the same as that of a KP system. In this case, however, the coefficient functions are constrained by (2.10). Therefore, we can think of the nonlinear Schrödinger equation as a constrained KP system [7,9,12].

We will next make some observations on this system which will be useful in our later discussions. First, let us note that given \tilde{L} , we can define its formal adjoint [17]

$$\mathcal{L} = \tilde{L}^* = -(\partial + \bar{q}\partial^{-1}q) \quad (2.11)$$

It is straight forward to check that the standard Lax equation

$$\frac{\partial \mathcal{L}}{\partial t} = [(\mathcal{L}^2)_+, \mathcal{L}] \quad (2.12)$$

also gives the nonlinear Schrödinger equation. Furthermore, we note that with the identification

$$\bar{q} = q \quad (2.13)$$

the standard Lax equation

$$\frac{\partial \tilde{L}}{\partial t} = [\tilde{L}, (\tilde{L}^3)_+] \quad (2.14)$$

leads to the mKdV equation (the signs and factors can be appropriately redefined by scaling of variables)

$$\frac{\partial q}{\partial t} = -(q''' + 6q^2q') \quad (2.15)$$

The Lax operator, \mathcal{L} , with the identification in (2.13) becomes

$$\mathcal{L} = -\tilde{L} \quad (2.16)$$

and also gives the mKdV equation as

$$\frac{\partial \mathcal{L}}{\partial t} = [(\mathcal{L}^3)_+, \mathcal{L}] \quad (2.17)$$

This shows that the mKdV equation can be embedded into the nonlinear Schrödinger equation and it appears from our discussion that the Lax operator and its formal adjoint yield equivalent results.

3. Super NSE As a Nonstandard Constrained Super KP System

We have shown in an earlier publication [13] that the supersymmetric two boson hierarchy can be represented in the superspace by the Lax operator

$$L = D^2 - (D\Phi_0) + D^{-1}\Phi_1 \quad (3.1)$$

where Φ_0 and Φ_1 are two fermionic superfields and D is the covariant derivative in the superspace of the form

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \quad (3.2)$$

The nonstandard Lax equation

$$\frac{\partial L}{\partial t} = \left[L, (L^2)_{\geq 1} \right] \quad (3.3)$$

leads to the supersymmetric generalization of (2.3), namely, (see ref. 12 for details)

$$\begin{aligned} \frac{\partial \Phi_0}{\partial t} &= -(D^4 \Phi_0) + 2(D\Phi_0)(D^2 \Phi_0) + 2(D^2 \Phi_1) \\ \frac{\partial \Phi_1}{\partial t} &= (D^4 \Phi_1) + 2(D^2((D\Phi_0)\Phi_1)) \end{aligned} \quad (3.4)$$

The system of equations (3.4) reduce to the supersymmetric nonlinear Schrödinger equation of the form

$$\begin{aligned} \frac{\partial Q}{\partial t} &= -(D^4 Q) + 2(D((DQ)\overline{Q}))Q \\ \frac{\partial \overline{Q}}{\partial t} &= (D^4 \overline{Q}) - 2(D((D\overline{Q})Q))\overline{Q} \end{aligned} \quad (3.5)$$

with the field identifications

$$\begin{aligned} \Phi_0 &= -D \ln(DQ) + D^{-1}(\overline{Q}Q) \\ \Phi_1 &= -\overline{Q}(DQ) \end{aligned} \quad (3.6)$$

Here Q and \overline{Q} are fermionic superfields.

Let us next consider the Lax operator (3.1) with the field identifications in (3.6) and note that

$$\begin{aligned} L &= D^2 + \frac{(D^3 Q)}{(DQ)} - \overline{Q}Q - D^{-1}\overline{Q}(DQ) \\ &= (DQ)^{-1} (D^2 - \overline{Q}Q - (DQ)D^{-1}\overline{Q}) (DQ) \\ &= G\tilde{L}G^{-1} \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} G &= (DQ)^{-1} \\ \tilde{L} &= D^2 - \overline{Q}Q - (DQ)D^{-1}\overline{Q} \end{aligned} \quad (3.8)$$

The two Lax operators, L and \tilde{L} , are related by a gauge transformation in superspace. This is very much like the bosonic case. However, unlike our earlier discussion, \tilde{L} does not

lead to any consistent equation in the standard or nonstandard representation of the Lax equation.

Let us next note that the formal adjoint of \tilde{L} in (3.8) can be written as

$$\mathcal{L} = \tilde{L}^* = - (D^2 + \overline{Q}Q - \overline{Q}D^{-1}(DQ)) \quad (3.9)$$

Through straight forward calculations, it can now be checked that the nonstandard Lax equation

$$\frac{\partial \mathcal{L}}{\partial t} = \left[\mathcal{L}, (\mathcal{L}^2)_{\geq 1} \right] \quad (3.10)$$

gives the supersymmetric nonlinear Schrödinger equations of (3.5). We see that there are two basic differences from the bosonic case discussed in sec. 2. First, it is the formal adjoint of the gauge transformed Lax operator which leads to consistent equations and second, the supersymmetric generalization of (2.5) is obtained as a nonstandard Lax equation.

We also note that we can write

$$\begin{aligned} \mathcal{L} &= - (D^2 + \overline{Q}Q - \overline{Q}D^{-1}(DQ)) \\ &= - (D^2 + \overline{Q}Q - \overline{Q}(DQ)D^{-1} - \overline{Q}(D^2Q)D^{-2} + \overline{Q}(D^3Q)D^{-3} + \dots) \\ &= - \left(D^2 + \sum_{n=-1}^{\infty} \Phi_n D^{-n} \right) \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} \Phi_{-1} &= 0 \\ \Phi_n &= (-1)^{[\frac{n+1}{2}]} \overline{Q}(D^n Q), \quad n \geq 0 \end{aligned} \quad (3.12)$$

Here Φ_{2n} (Φ_{2n+1}) are bosonic (fermionic) superfields and $[n/2]$ stands for the integral part of $n/2$. Also, in (3.11) we have used the generalized Liebnitz rule given in ref. 18. There are several points to emphasize here. First, we note that the form of \mathcal{L} in (3.11) is identical to that of the Lax operator for susy KP. However, it is an even parity Lax operator [19] and not of the Manin-Radul type [18] (our result, therefore, differs from the one in ref. 20). Second, the coefficient superfields Φ_n are constrained by (3.12). And finally, the Lax equation that gives the supersymmetric nonlinear Schrödinger equation, namely (3.10), is a nonstandard type. Therefore, we can think of the supersymmetric nonlinear Schrödinger equation as a nonstandard, constrained supersymmetric KP system.

From the structure of the Lax equation in (3.10), one can show that the conserved quantities of the system are given by

$$H^{(n)} = \frac{1}{n} \int dx d\theta \text{sRes } \mathcal{L}^n = \frac{1}{n} \int d\mu \text{sRes } \mathcal{L}^n \quad (3.13)$$

where “sRes” stands for the super residue which is defined to be the coefficient of D^{-1} (D^{-1} is assumed to be on the right). The first few conserved quantities have the form

$$\begin{aligned} H^{(1)} &= \int d\mu \overline{Q}(DQ) \\ H^{(2)} &= \frac{1}{2} \int d\mu (\overline{Q}(D^3Q) - (D^3\overline{Q})Q) \\ H^{(3)} &= -\frac{1}{2} \int d\mu \left[(D^3\overline{Q})(D^2Q) + (D^2\overline{Q})(D^3Q) \right. \\ &\quad \left. + (D\overline{Q})(DQ) ((D\overline{Q})Q + \overline{Q}(DQ)) \right. \\ &\quad \left. - \overline{Q}Q ((D^2\overline{Q})(DQ) - (D\overline{Q})(D^2Q)) \right] \end{aligned} \quad (3.14)$$

These can be compared with the conserved quantities of ref. 14 for $k = 1$. It can also be checked that the system of equations (3.5) are Hamiltonian with respect to $H^{(2)}$ and the Hamiltonian structure

$$\begin{aligned} \{Q(x_1, \theta_1, t), Q(x_2, \theta_2, t)\} &= Q(x_1, \theta_1, t)Q(x_2, \theta_2, t)D_1^{-1}\Delta_{12} \\ \{Q(x_1, \theta_1, t), \overline{Q}(x_2, \theta_2, t)\} &= -\frac{1}{2}D_1\Delta_{12} - Q(x_1, \theta_1, t)\overline{Q}(x_2, \theta_2, t)D_1^{-1}\Delta_{12} \\ \{\overline{Q}(x_1, \theta_1, t), \overline{Q}(x_2, \theta_2, t)\} &= \overline{Q}(x_1, \theta_1, t)\overline{Q}(x_2, \theta_2, t)D_1^{-1}\Delta_{12} \end{aligned} \quad (3.15)$$

where

$$\Delta_{12} = \delta(x_1 - x_2)\delta(\theta_1 - \theta_2) \quad (3.16)$$

To conclude this section, let us note that if we identify

$$\overline{Q} = Q \quad (3.17)$$

then

$$\mathcal{L} = -(D^2 - QD^{-1}(DQ)) \quad (3.18)$$

and unlike the bosonic case, it is different from its formal adjoint with the same identification. It can also be checked that with the identification in (3.17), the nonstandard Lax equation

$$\frac{\partial \mathcal{L}}{\partial t} = \left[\mathcal{L}, (\mathcal{L}^3)_{\geq 1} \right] \quad (3.19)$$

yields

$$\frac{\partial Q}{\partial t} = -(D^6 Q) + 3 (D^2 (Q(DQ))) (DQ) \quad (3.20)$$

This is nothing other than the supersymmetric mKdV equation [21] and this shows how the susy mKdV equation can be embedded into the susy nonlinear Schrödinger equation in a nonstandard Lax representation. This is quite analogous to the embedding of the supersymmetric KdV equation in the supersymmetric two boson hierarchy (see ref. 13 for details).

4. General Flows of the Nonstandard Super KP Hierarchy

Let us consider a general super Lax operator of the form (3.11)

$$\begin{aligned} L &= D^2 + \Phi_0 + \Phi_1 D^{-1} + \Phi_2 D^{-2} + \dots \\ &= D^2 + \sum_{n=0}^{\infty} \Phi_n D^{-n} \end{aligned} \quad (4.1)$$

where the Grassmann parity of the superfields Φ_n are

$$|\Phi_n| = \frac{1 - (-1)^n}{2} \quad (4.2)$$

Let the expansion of the superfields be of the form

$$\begin{aligned} \Phi_{2n} &= q_{2n} + \theta \phi_{2n} \\ \Phi_{2n+1} &= \phi_{2n+1} + \theta q_{2n+1} \end{aligned} \quad (4.3)$$

where q_n (ϕ_n) are the bosonic (fermionic) components of the superfields.

The nonstandard flows associated with this super KP Lax operator are given by

$$\frac{\partial L}{\partial t_n} = \left[(L^n)_{\geq 1}, L \right] \quad (4.4)$$

For $n = 1$, the flow is quite trivial and gives

$$\frac{\partial \Phi_n}{\partial t_1} = (D^2 \Phi_n) = \left(\frac{\partial \Phi_n}{\partial x} \right) \quad (4.5)$$

This implies that the time coordinate t_1 can be identified with x .

For $n = 2$, the flow in (4.4) gives

$$\begin{aligned} \frac{\partial \Phi_n}{\partial t_2} = & (D^4 \Phi_n) + 2(D^2 \Phi_{n+2}) + 2\Phi_0(D^2 \Phi_n) + 2\Phi_1(D \Phi_n) - 2(1 + (-1)^n) \Phi_1 \Phi_{n+1} \\ & + 2 \sum_{\ell \geq 1} \left\{ -(-1)^{[\ell/2]} \begin{bmatrix} n+1 \\ \ell \end{bmatrix} \Phi_{n-\ell+2} (D^\ell \Phi_0) + (-1)^{[\ell/2]+n} \begin{bmatrix} n \\ \ell \end{bmatrix} \Phi_{n-\ell+1} (D^\ell \Phi_1) \right\} \end{aligned} \quad (4.6)$$

where the super binomial coefficients $\begin{bmatrix} n \\ \ell \end{bmatrix}$ are defined in ref. 18. The equations for the bosonic components can be obtained from (4.6) to be

$$\begin{aligned} \frac{\partial q_{2n}}{\partial t_2} = & q_{2n}'' + 2q_{2n+2}' + 2q_0 q_{2n}' + 2\phi_1 \phi_{2n} - 4\phi_1 \phi_{2n+1} \\ & + 2 \sum_{\ell \geq 1} (-1)^\ell \left\{ - \begin{bmatrix} 2n+1 \\ 2\ell \end{bmatrix} q_{2n-2\ell+2} q_0^{(\ell)} + \begin{bmatrix} 2n+1 \\ 2\ell-1 \end{bmatrix} \phi_{2n-2\ell+3} \phi_0^{(\ell-1)} \right. \\ & \left. + \begin{bmatrix} 2n \\ 2\ell \end{bmatrix} \phi_{2n-2\ell+1} \phi_1^{(\ell)} - \begin{bmatrix} 2n \\ 2\ell-1 \end{bmatrix} q_{2n-2\ell+2} q_1^{(\ell-1)} \right\} \end{aligned} \quad (4.7)$$

$$\begin{aligned} \frac{\partial q_{2n+1}}{\partial t_2} = & q_{2n+1}'' + 2q_{2n+3}' + 2(\phi_0 \phi_{2n+1}' + q_0 q_{2n+1}') + 2(q_1 q_{2n+1} - \phi_1 \phi_{2n+1}') \\ & + 2 \sum_{\ell \geq 1} (-1)^\ell \left\{ - \begin{bmatrix} 2n+2 \\ 2\ell \end{bmatrix} (-\phi_{2n-2\ell+3} \phi_0^{(\ell)} + q_{2n-2\ell+3} q_0^{(\ell)}) \right. \\ & + \begin{bmatrix} 2n+2 \\ 2\ell-1 \end{bmatrix} (\phi_{2n-2\ell+4} \phi_0^{(\ell-1)} + q_{2n-2\ell+4} q_0^{(\ell)}) \\ & - \begin{bmatrix} 2n+1 \\ 2\ell \end{bmatrix} (\phi_{2n-2\ell+2} \phi_1^{(\ell)} + q_{2n-2\ell+2} q_1^{(\ell)}) \\ & \left. + \begin{bmatrix} 2n+1 \\ 2\ell-1 \end{bmatrix} (q_{2n-2\ell+3} q_1^{(\ell-1)} - \phi_{2n-2\ell+3} \phi_1^{(\ell)}) \right\} \end{aligned} \quad (4.8)$$

In the bosonic limit – when all the ϕ_n 's are zero – we note that if we set

$$q_{2n} = 0, \quad \text{for all } n \quad (4.9)$$

and identify

$$q_{2n+1} = u_n, \quad \text{for all } n \quad (4.10)$$

then (4.8) gives

$$\begin{aligned} \frac{\partial u_0}{\partial t_2} &= u_0'' + 2u_1' \\ \frac{\partial u_1}{\partial t_2} &= u_1'' + 2u_2' + 2u_0u_0' \\ \frac{\partial u_2}{\partial t_2} &= u_2'' + 2u_3' - 2u_0u_0'' + 4u_1u_0' \\ &\vdots \end{aligned} \quad (4.11)$$

which are nothing other than the t_2 -flows for the standard KP hierarchy [16,22].

On the other hand, in the bosonic limit, if we set

$$q_{2n+1} = 0, \quad \text{for all } n \quad (4.12)$$

and identify

$$q_{2n} = u_n, \quad \text{for all } n \quad (4.13)$$

then, (4.7) gives

$$\begin{aligned} \frac{\partial u_0}{\partial t_2} &= u_0'' + 2u_1' + 2u_0u_0' \\ \frac{\partial u_1}{\partial t_2} &= u_1'' + 2u_2' + 2u_0u_1' + 2u_0'u_1 \\ \frac{\partial u_2}{\partial t_2} &= u_2'' + 2u_3' - 2u_1u_0'' + 2u_0u_2' + 4u_2u_0' \\ &\vdots \end{aligned} \quad (4.14)$$

which are nothing other than the t_2 -flows associated with the mKP hierarchy [16,22].

For $n = 3$, equation (4.4) gives

$$\begin{aligned}
\frac{\partial \Phi_n}{\partial t_3} = & (D^6 \Phi_n) + 3(D^4 \Phi_{n+2}) + 3(D^2 \Phi_{n+4}) + 3\Phi_0(D^4 \Phi_n) + 6\Phi_0(D^2 \Phi_{n+2}) \\
& + 3\Phi_1(D^3 \Phi_n) + 3\Phi_1(D \Phi_{n+2}) - 3(-1)^n \Phi_1(D^2 \Phi_{n+1}) \\
& - 3(1 + (-1)^n) \Phi_1 \Phi_{n+3} - 3(1 + (-1)^n) ((D^2 \Phi_1) + \Phi_3 + 2\Phi_1 \Phi_0) \Phi_{n+1} \\
& + 3((D^2 \Phi_0) + \Phi_2 + \Phi_0^2)(D^2 \Phi_n) + 3((D^2 \Phi_1) + \Phi_3 + 2\Phi_1 \Phi_0)(D \Phi_n) \\
& + 3 \sum_{\ell \geq 1} \left\{ -(-1)^{[\ell/2]} \begin{bmatrix} n+3 \\ \ell \end{bmatrix} \Phi_{n-\ell+4}(D^\ell \Phi_0) \right. \\
& \quad + (-1)^{[\ell/2]+n} \begin{bmatrix} n+2 \\ \ell \end{bmatrix} \Phi_{n-\ell+3}(D^\ell \Phi_1) \\
& \quad + (-1)^{[\ell/2]} \begin{bmatrix} n+1 \\ \ell \end{bmatrix} \Phi_{n-\ell+2}(D^\ell ((D^2 \Phi_0) + \Phi_2 + \Phi_0^2)) \\
& \quad \left. + (-1)^{[\ell/2]+n} \begin{bmatrix} n \\ \ell \end{bmatrix} \Phi_{n-\ell+1}(D^\ell ((D^2 \Phi_1) + \Phi_3 + 2\Phi_1 \Phi_0)) \right\}
\end{aligned} \tag{4.15}$$

The bosonic components can again be obtained from (4.15) and they have the form

$$\begin{aligned}
\frac{\partial q_{2n}}{\partial t_3} = & q_{2n}''' + 3q_{2n+2}'' + 3q_{2n+4}' + 3q_0 q_{2n}'' + 6q_0 q_{2n+2}' + 3\phi_1 \phi_{2n}' \\
& + 3\phi_1 \phi_{2n+2} - 3\phi_1 \phi_{2n+1}' - 6\phi_1 \phi_{2n+3} - 6(\phi_1' + 2q_0 \phi_1 + \phi_3) \phi_{2n+1} \\
& + 3(q_0^2 + q_0' + q_2) q_{2n}' + 3(\phi_1' + 2q_0 \phi_1 + \phi_3) \phi_{2n} \\
& + 3 \sum_{\ell \geq 1} (-1)^\ell \left\{ - \begin{bmatrix} 2n+3 \\ 2\ell \end{bmatrix} q_{2n-2\ell+4} q_0^{(\ell)} + \begin{bmatrix} 2n+3 \\ 2\ell-1 \end{bmatrix} \phi_{2n-2\ell+5} \phi_0^{(\ell-1)} \right. \\
& \quad + \begin{bmatrix} 2n+2 \\ 2\ell \end{bmatrix} \phi_{2n-2\ell+3} \phi_1^{(\ell)} - \begin{bmatrix} 2n+2 \\ 2\ell-1 \end{bmatrix} q_{2n-2\ell+4} q_1^{(\ell-1)} \\
& \quad - \begin{bmatrix} 2n+1 \\ 2\ell \end{bmatrix} q_{2n-2\ell+2} (q_0^2 + q_0' + q_2)^{(\ell)} \\
& \quad + \begin{bmatrix} 2n+1 \\ 2\ell-1 \end{bmatrix} \phi_{2n-2\ell+3} (2q_0 \phi_0 + \phi_0' + \phi_2)^{(\ell-1)} \\
& \quad + \begin{bmatrix} 2n \\ 2\ell \end{bmatrix} \phi_{2n-2\ell+1} (\phi_1' + 2q_0 \phi_1 + \phi_3)^{(\ell)} \\
& \quad \left. - \begin{bmatrix} 2n \\ 2\ell-1 \end{bmatrix} q_{2n-2\ell+2} (q_1' + 2q_0 q_1 + 2\phi_0 \phi_1 + q_3)^{(\ell-1)} \right\}
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
\frac{\partial q_{2n+1}}{\partial t_3} = & q_{2n+1}''' + 3q_{2n+3}'' + 3q_{2n+5}' + 3(q_0 q_{2n+1}'' + \phi_0 \phi_{2n+1}'') \\
& + 6(q_0 q_{2n+3}' + \phi_0 \phi_{2n+3}') + 3(q_1 q_{2n+1}'' - \phi_1 \phi_{2n+1}'') \\
& + 3(q_1 q_{2n+3} - \phi_1 \phi_{2n+3}') + 3(q_1 q_{2n+2}' - \phi_1 \phi_{2n+2}') \\
& + 3(2q_0 \phi_0 + \phi_0' + \phi_2) \phi_{2n+1}' + 3(q_0^2 + q_0' + q_2) q_{2n+1}' \\
& + 3(q_1' + 2q_0 q_1 + 2\phi_0 \phi_1 + q_3) q_{2n+1} - 3(\phi_1' + 2q_0 \phi_1 + 2q_1 \phi_0 + \phi_3) \phi_{2n+1}' \\
& + 3 \sum_{\ell \geq 1} (-1)^\ell \left\{ - \begin{bmatrix} 2n+4 \\ 2\ell \end{bmatrix} (q_{2n-2\ell+5} q_0^{(\ell)} - \phi_{2n-2\ell+5} \phi_0^{(\ell)}) \right. \\
& + \begin{bmatrix} 2n+4 \\ 2\ell-1 \end{bmatrix} (\phi_{2n-2\ell+6} \phi_0^{(\ell-1)} + q_{2n-2\ell+6} q_0^{(\ell)}) \\
& - \begin{bmatrix} 2n+3 \\ 2\ell \end{bmatrix} (\phi_{2n-2\ell+4} \phi_1^{(\ell)} + q_{2n-2\ell+4} q_1^{(\ell)}) \\
& + \begin{bmatrix} 2n+3 \\ 2\ell-1 \end{bmatrix} (q_{2n-2\ell+5} q_1^{(\ell-1)} - \phi_{2n-2\ell+5} \phi_1^{(\ell)}) \\
& - \begin{bmatrix} 2n+2 \\ 2\ell \end{bmatrix} \left(q_{2n-2\ell+3} (q_0^2 + q_0' + q_2)^{(\ell)} \right. \\
& \quad \left. - \phi_{2n-2\ell+3} (2q_0 \phi_0 + \phi_0' + \phi_2)^{(\ell)} \right) \\
& + \begin{bmatrix} 2n+2 \\ 2\ell-1 \end{bmatrix} \left(\phi_{2n-2\ell+4} (2q_0 \phi_0 + \phi_0' + \phi_2)^{(\ell-1)} \right. \\
& \quad \left. + q_{2n-2\ell+4} (q_0^2 + q_0' + q_2)^{(\ell)} \right) \\
& - \begin{bmatrix} 2n+1 \\ 2\ell \end{bmatrix} \left(\phi_{2n-2\ell+2} (\phi_1' + 2q_0 \phi_1 + 2q_1 \phi_0 + \phi_3)^{(\ell)} \right. \\
& \quad \left. + q_{2n-2\ell+2} (q_1' + 2q_0 q_1 + 2\phi_0 \phi_1 + q_3)^{(\ell)} \right) \\
& \left. + \begin{bmatrix} 2n+1 \\ 2\ell-1 \end{bmatrix} \left(q_{2n-2\ell+3} (q_1' + 2q_0 q_1 + 2\phi_0 \phi_1 + q_3)^{(\ell-1)} \right. \right. \\
& \quad \left. \left. - \phi_{2n-2\ell+3} (\phi_1' + 2q_0 \phi_1 + 2q_1 \phi_0 + \phi_3)^{(\ell)} \right) \right\} \quad (4.17)
\end{aligned}$$

We note here for completeness that, in the bosonic limit, with the identifications in (4.9) and (4.10), we obtain from (4.17)

$$\begin{aligned}
\frac{\partial u_0}{\partial t_3} &= u_0''' + 3u_1'' + 3u_2' + 6u_0u_0' \\
\frac{\partial u_1}{\partial t_3} &= u_1''' + 3u_2'' + 3u_3' + 6u_0u_1' + 6u_0'u_1 \\
&\vdots
\end{aligned} \tag{4.18}$$

which are the t_3 -flows for the standard KP hierarchy [16,22]. On the other hand, the identifications in (4.12) and (4.13) lead to (from (4.16))

$$\begin{aligned}
\frac{\partial u_0}{\partial t_3} &= u_0''' + 3u_1'' + 3u_2' + 3u_0u_0'' + 3(u_0')^2 + 6(u_1u_0)' + 3u_0^2u_0' \\
&\vdots
\end{aligned} \tag{4.19}$$

These are the t_3 -flows for the mKP hierarchy [16,22]. It is interesting to note that the nonstandard KP equation (4.4) contains both the standard KP and the mKP flows in its bosonic limit.

5. A New Super KP Equation

As we have shown in the last section, the nonstandard KP equation of (4.4) reduces to the standard KP flows in the bosonic limit. It is, therefore, interesting to examine in some detail the nature of the super KP equation that it leads to. To that end, we assume

$$\Phi_{2n} = 0, \quad \text{for all } n \tag{5.1}$$

The first two nontrivial equations following from (4.6) with this identifications are

$$\begin{aligned}
\frac{\partial \Phi_1}{\partial t_2} &= (D^4 \Phi_1) + 2(D^2 \Phi_3) \\
\frac{\partial \Phi_3}{\partial t_2} &= (D^4 \Phi_3) + 2(D^2 \Phi_5) - 2(D(\Phi_1 \Phi_3)) + 2\Phi_1(D^3 \Phi_1)
\end{aligned} \tag{5.2}$$

Similarly, the first nontrivial equation following from (4.15) with the identification in (5.1) has the form

$$\frac{\partial \Phi_1}{\partial t_3} = (D^6 \Phi_1) + 3(D^4 \Phi_3) + 3(D^2 \Phi_5) + 3(D^2(\Phi_1(D\Phi_1))) \tag{5.3}$$

From (5.2) and (5.3), we obtain

$$D^2 \left(\frac{\partial \Phi_1}{\partial t_3} - \frac{1}{4}(D^6 \Phi_1) - \frac{3}{2}(D^2(\Phi_1(D\Phi_1))) - \frac{3}{2}(D(\Phi_1(D^{-2} \frac{\partial \Phi_1}{\partial t_2}))) \right) = \frac{3}{4} \frac{\partial^2 \Phi_1}{\partial t_2^2} \tag{5.4}$$

With the identifications

$$t_2 = y, \quad t_3 = t \quad \text{and} \quad \Phi_1 = \Phi = \phi + \theta u \quad (5.5)$$

equation (5.4) becomes

$$D^2 \left(\frac{\partial \Phi}{\partial t} - \frac{1}{4}(D^6 \Phi) - \frac{3}{2}(D^2(\Phi(D\Phi))) - \frac{3}{2}(D(\Phi(\partial^{-1} \frac{\partial \Phi}{\partial y}))) \right) = \frac{3}{4} \frac{\partial^2 \Phi}{\partial y^2} \quad (5.6)$$

It is now straight forward to check that (5.6) reduces in the bosonic limit to

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} - \frac{1}{4}u''' - 3uu' \right) = \frac{3}{4} \frac{\partial^2 u}{\partial y^2} \quad (5.7)$$

which is the KP equation. The supersymmetric generalization in (5.6), however, differs from the Manin-Radul equation [18,23] because of the presence of the nonlocal terms. We note that in components (5.6) takes the form

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} - \frac{1}{4}u''' - 3uu' + \frac{3}{2}\phi\phi'' - \frac{3}{2}\phi'(\partial^{-1} \frac{\partial \phi}{\partial y}) - \frac{3}{2}\phi \frac{\partial \phi}{\partial y} \right) &= \frac{3}{4} \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial t} - \frac{1}{4}\phi''' - \frac{3}{2}(u\phi)' - \frac{3}{2}u(\partial^{-1} \frac{\partial \phi}{\partial y}) + \frac{3}{2}\phi(\partial^{-1} \frac{\partial u}{\partial y}) \right) &= \frac{3}{4} \frac{\partial^2 \phi}{\partial y^2} \end{aligned} \quad (5.8)$$

These equations are not invariant under

$$y \leftrightarrow -y \quad (5.9)$$

unlike the Manin-Radul equations. However, we note that when we restrict the variables u and ϕ to be independent of y , these equations reduce to the supersymmetric KdV equation [21]. These, therefore, represent a new supersymmetric generalization of the KP equation.

6. Conclusion

We have shown that the supersymmetric nonlinear Schrödinger equation can be represented as a nonstandard, constrained super KP flow. We have constructed the conserved quantities of the system in this formalism and we have shown how the supersymmetric mKdV equation can be embedded into this system. We have worked out the first three flows associated with a general, nonstandard, super KP system and we have shown that

these flows contain both the standard KP flows as well as the mKP flows in their bosonic limit. We have shown that these flows lead to a new supersymmetrization of the KP equation that is nonlocal. It has the correct bosonic limit and when properly restricted, it reduces to the supersymmetric KdV equation. However, this equation is different from the Manin-Radul equation because of nonlocal terms which are also antisymmetric under $y \leftrightarrow -y$. Properties of this system are under study and will be reported in a later publication.

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